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The Scalar Riemann Problem in Two Spatial Dimensions: Piecewise Smoothness of Solutions and its Breakdown

W. B. Lindquist

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in two spatial dimensions

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The Scalar Riemann Problem in Two Spatial Dimensions: Piecewise Smoothness of Solutions and its Breakdown

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ABSTRACT

Consider the scalar quasilinear equation

$$\frac{\partial u(t,x)}{\partial t} + \sum_{i=1}^{n} \frac{\partial f_i(u(t,x))}{\partial x_i} = 0 ,$$

for $n = 1, 2$, $f_i \in C^2 : R \to R$. For $n = 2$, we define the two dimensional Riemann problem and show the unique (in the sense of Kruzkov) solutions are piecewise smooth for $f_1 = f_2 = f$, f purely convex or having a single inflection point. A mechanism leading to a presumed loss of piecewise smoothness is presented for f having three or more inflection points. The analysis is based on a study of the generalization of the one dimensional Riemann problem to allow for initial data having a finite number of jump discontinuities with constant data or rarefaction waves between jumps.

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1. Introduction

Consider the Cauchy problem for the scalar quasilinear equation

$$\frac{\partial u(t,x)}{\partial t} + \sum_{i=1}^{n} \frac{\partial f_i(u(t,x))}{\partial x_i} = 0 , \qquad n = 1, 2 ,$$

$$u(0,x) = u_0(x) , \tag{1.1}$$

with $f_i: R \to R \in C^2$. For two spatial dimensions, we define the Riemann problem and show the unique (in the sense of Kruzkov) solutions are piecewise smooth for $f_1 = f_2 = f$, f having at most one inflection point. The assumption $f_1 = f_2$ appears to be natural from the point of view of applications. This sufficiency condition follows from a detailed analysis of a generalization of the Riemann problem in one dimension to allow for initial data having a finite number of jump discontintuities where the data between points of jump is allowed to contain rarefaction waves as defined below. For these generalized one dimensional Riemann problems, we show the unique (in the sense of Kruzkov) solutions are piecewise smooth if the initial data has a single jump discontinuity and f_1 is restricted to at most two inflection points. If the initial data has more than a single jump discontinuity we show that restricting f_1 to at most one inflection point guarantees a piecewise smooth solution.

We present a construction based on a function f_1 having three inflection points which shows a possible breakdown of piecewise smoothness for the associated one dimensional Riemann problem. The implication of this example is that the scalar Riemann problem in two dimensions can presumably fail to have piecewise smooth solutions for non convex $f_2 = f_1$ having three inflection points.

A bounded measurable function $u(t,x)$ is a *weak solution* of (1.1) in the strip $\Pi_T = [0,T] \times R^n$ if

$$\int\int_{\Pi_T} dt\, dx \left[u \frac{\partial \phi}{\partial t} + \sum_{i=1}^{2} f_i(u) \frac{\partial \phi}{\partial x_i} \right] + \int_{t=0} dx\, u_0(x)\, \phi(0,x) = 0 , \tag{1.2}$$

for all $\phi \in C_0^\infty(\Pi_T)$. A function is a weak solution in the large if it is a weak solution for all T. For initial data $u_0(x)$ in the class of bounded, measurable functions, Kruzkov [2] has shown the existence and uniqueness of a bounded measurable function which is a weak solution to (1.1)[1].

[1] Kruskov's proof holds for $n \geq 1$ spatial dimensions. Here only $n \leq 2$ concerns us.

The uniqueness (entropy) condition satisfied by this weak solution is [2]

$$\int_{\Pi_T}\int dt\,dx\ \mathrm{sign}(u-k)\left\{ (u-k)\frac{\partial\phi}{\partial t} + \sum_{i=1}^{2} (f_i(u) - f_i(k))\frac{\partial\phi}{\partial x_i} \right\} \geq 0\,, \tag{1.3}$$

for any real constant k and any $\phi \in C_0^\infty(\Pi_T)$ such that $\phi \geq 0$. This uniqueness condition characterizes the allowed discontinuities in the weak solution.

Define a *regular point* of the weak solution u as a point p for which $\partial u/\partial t$ and $\partial u/\partial x_i$ exist and are continuous in a neighbourhood of p, and at which the differential form of (1.1) is satisfied. The function u shall be termed *smooth* in an open neighbourhood $\mathcal{N}$ if each point $p \in \mathcal{N}$ is a regular point. Let a *smooth* surface of codimension i in $R^n \times R^+$ be a surface at each point p of which a unique (modulo minus signs) i-dimensional space of normals exist and for which the normal space varies continuously as the point p is varied over the surface. A *piecewise smooth surface* is defined by induction on the codimension i, as the union of a finite number of smooth surfaces with piecewise smooth boundaries of codimension $i+1$; each of the finite number of pieces being everywhere the boundary of the same codimension i surfaces. A *piecewise smooth* solution $u \in R^n \times R^+$ is defined as a function that is smooth except on a discontinuity set that consists of piecewise smooth surfaces. A *point of jump* is a point p of discontinuity of a function u for which the discontinuity is a simple jump in u - ie. locally the point p lies on a smooth codimension 1 surface of discontinuity and the one sided limits $u^- = \lim_{\epsilon \to 0^+} u(p - \epsilon n)$ and $u^+ = \lim_{\epsilon \to 0^+} u(p + \epsilon n)$, where n is the unit normal to the smooth discontinuity surface at p, exist at p. A singular point of u which is not a point of jump shall be termed an *irregular* point.

Using integration by parts, the arbitrariness of ϕ and appropriate choices of the constant k, it can be shown that (1.3) is equivalent to two conditions for points of jump p in whose neighbourhood the unique weak solution u is piecewise smooth:

$$n \cdot [\, u^+ - u^-,\ \vec{f}(u^+) - \vec{f}(u^-)\,] = 0, \tag{1.4}$$

and

$$n \cdot [\, k - u^-,\ \vec{f}(k) - \vec{f}(u^-)\,] \geq 0\,, \tag{1.5}$$

where $\vec{f}$ stands for the vector $[\, f_1, f_2\,]$. In (1.5) n is oriented such that $u^- \leq u^+$ and k is any constant such that $u^- \leq k \leq u^+$. As a consequence of (1.4), (1.5) also holds with u^- replaced by

u^+. Equation (1.4) will be denoted the jump condition for the discontinuity at p and (1.5) the entropy condition. In $R^1 \times R^+$ (1.4) is the familiar Rankine-Hugoniot condition and (1.5) is equivalent to the entropy condition of Oleinik [5]. The advantage of equations (1.4) and (1.5) over (1.3) is that the former are local conditions which will allow construction of piecewise smooth global solutions by piecing together local solutions which individually obey (1.4) and (1.5).

A weak solution $u \in R^n \times R^+$ to (1.1) which is piecewise smooth and whose discontinuity sets are piecewise smooth surfaces of points of jump obeying (1.4) and (1.5) then satisfies (1.3) and is therefore the unique Kruzkov solution to (1.1).

The existence and uniqueness theory gives little insight into the actual form of the weak solution. In $R^1 \times R^+$, insight is provided by analysis of the Riemann problem whose solution is piecewise smooth and can be derived in terms of a non-linear wave analysis. It can be shown (see, for example, Lax [3] and the references therein) that these non-linear waves consist of rarefaction and shock waves. The initial motivation of this paper was to consider the natural generalization of the Riemann problem in two dimensions and investigate its solution in terms of two dimensional waves which, by implication, is a restriction to the set of piecewise smooth solutions. In this paper we determine a sufficient condition for which the solution to the two dimensional Riemann problem is piecewise smooth. In a companion paper [4] we investigate the form of these solutions in terms of non-linear waves. For the case $f_1 = f_2 = f$, the sufficiency condition consists of restricting f to at most one inflection point. We conjecture that if the number of inflection points of f is at least three, the solution to the generalized Riemann problem defined in 1 dimension (and hence for the Riemann problem in two dimensions) may fail to be piecewise smooth. Our argument for this is presented in §2.

The generalizations of the 1-dimensional Riemann problem to allow multiple jumps and restricted smooth variation in the initial data are presented in §§2 and 3 . In §4, the two dimensional Riemann problem is defined and a sufficiency condition for piecewise smooth solutions obtained.

2. The 1-Dimensional Riemann Problem

Throughout this paper the notations $q'(p)$ or $(q(p))'$ shall denote $d\,q(p)\,/\,dp$ for any function q of a single variable p.

We first discuss the familiar results for the one dimensional Riemann problem to introduce notation and reformulate results in a manner that will allow generalization. The scalar quasilinear equation (1.1) in one spatial dimension ($f_1 = f,\ x_1 = x$) with initial data

$$u(0, x) = \begin{cases} u_l \text{ for } x < x_0\,, \\ u_r \text{ for } x > x_0\,. \end{cases} \tag{2.1}$$

defines the *1-dimensional Riemann problem.*

The Riemann problem (1.1), (2.1) is invariant under the similarity transformation $(t, x - x_0) \to (ct, c(x - x_0)),\ c > 0$.[2] Hence the solution is constant along the straight lines $(x - x_0)/t = constant$. In a neighbourhood of a point in which the Riemann problem solution is smooth, (1.1) implies that the solution will be constant along the straight lines

$$\frac{x - x_c}{t - t_c} = f'(u)\,, \quad t \geq t_c\,, \tag{2.2}$$

which are denoted *characteristics*. Characteristics shall be denoted (u, t_c, x_c) where (t_c, x_c) is the origin point of the characteristic given by (2.2). For the Riemann problem (1.1) (2.1), $t_c = 0$ for all characteristics.

Let $p = (t_p, x_p)$ be a point on a characteristic (u_1, t_1, x_1). Let $\hat{n}$ be a normal to the characteristic at p. By the smoothness of u there exists δ_p such that each point $p + \epsilon\,\hat{n}$, where $-\delta_p \leq \epsilon \leq \delta_p$, lies on a characteristic. Let (u_e, t_e, x_e) denote such a characteristic. Then (u_1, t_1, x_1) is a *rarefaction wave* if the lines defined by the characteristics (u_1, t_1, x_1) and (u_e, t_e, x_e) intersect at some $t \leq t_p$. This condition generalizes the concept of the familiar centered rarefaction wave for which $t_e = t_1$, $x_e = x_1$, to non-centered rarefaction waves which appear in the generalized Riemann problem solutions and also generalizes to a definition of rarefaction waves in two spatial dimensions.

[2] This scale invariance is one of the essential features of the Riemann problem.

A *rarefaction fan* is an open set in the t, x plane all points of which belong to rarefaction waves.

A *shock* is defined as a **smooth** curve of points of jump in the t, x plane at each point of which the conditions (1.4) and (1.5) are satisfied. A shock curve (wave) will be denoted $(u_l(t), u_r(t), t_s, x_s)$ where $u_l(t)$ is the limit value of the solution u to the left of the shock wave at time $t > t_s$, u_r is the corresponding right limit, and (t_s, x_s) is the origin point of the shock wave. For the Riemann problem (1.1) (2.1), $t_s = 0$, $x_s = x_0$.

A *constant state* is a domain (connected open set) in the t, x plane over which the solution is constant.

Let $f(u) \in C^2: R \to R$. Let $\{u_1, u_2\}$ denote a closed interval where

$$\{u_1, u_2\} = \begin{cases} [u_1, u_2] & \text{if } u_1 < u_2 , \\ [u_2, u_1] & \text{if } u_1 > u_2 . \end{cases} \tag{2.3}$$

A *rarefaction interval of* f shall denote an interval which satisfies either

$$f''(u) > 0 \quad \text{if } u_1 < u_2 \quad \text{for every } u \in \{u_1, u_2\} , \tag{2.4a}$$

or

$$f''(u) < 0 \quad \text{if } u_1 > u_2 \quad \text{for every } u \in \{u_1, u_2\} . \tag{2.4b}$$

Let $\mathcal{E}(u_1, u_2; u): R \to R$ denote the convex envelope function for $f(u)$ restricted to the interval $\{u_1, u_2\}$ of f such that

$$\mathcal{E}(u_1, u_2; u) = \begin{cases} \text{upper convex envelope from } u_1 \text{ to } u_2 , & \text{if } u_1 > u_2 \\ \text{lower convex envelope from } u_1 \text{ to } u_2 , & \text{if } u_1 < u_2 \end{cases} \tag{2.5}$$

Let $C(u_1, u_2; u): R \to R$ denote the linear map restricted to the domain interval $\{u_1, u_2\}$ such that

$$\begin{aligned} C(u_1, u_2; u_1) &= f(u_1) , \\ C(u_1, u_2; u_2) &= f(u_2) , \end{aligned} \tag{2.6}$$

i.e. for all u in $\{u_1, u_2\}$,

$$\frac{d\, C(u_1, u_2; u)}{du} = \frac{f(u_2) - f(u_1)}{u_2 - u_1} .$$

Denote the restriction of the function $f(u)$ to the sub-interval $\{u_1, u_2\}$ by $f(u_1, u_2; u)$.

Theorem 2.1: (Oleinik [5]) Consider the Riemann problem (1.1), (2.1). Assume $f(u) \in C^2 : R \to R$ such that f has a finite number of inflection points. Then the unique, entropy obeying solution connecting the states u_l and u_r is determined solely from $\mathcal{E}(u_l, u_r; u)$, as described below. Further, the solution is piecewise smooth, with a single irregular point corresponding to the discontinuity at x_0 in the initial data.

The convex envelope function $\mathcal{E}(u_l, u_r; u)$ uniquely divides the interval $\{u_l, u_r\}$ into rarefaction intervals separated by intervals corresponding to the linear chord segments of $\mathcal{E}$. Each rarefaction interval corresponds to a rarefaction fan in the solution. The fans are centered on $t = 0, x = x_0$ and each rarefaction wave has a unique value u corresponding to a state in a rarefaction interval. The slope of the rarefaction wave $(v, 0, x_0)$ is given by $f'(v)$. Each linear chord in $\mathcal{E}$ covering an interval $\{u_1, u_2\}$ corresponds to a shock wave $(u_1, u_2, 0, x_0)$. The solution is bordered on the left by the single constant state $u = u_l$ and on the right by $u = u_r$. No constant states appear within the solution except for these two bounding constant states, which are the states u_l and u_r of the initial data. The speeds of the non-linear waves found in the solution increase continuously from left to right along the x direction. Fig. 2.1a shows an upper convex envelope drawn between two states for the function $f(u)$ illustrated. Fig. 2.1b illustrates the corresponding Riemann solution in t, x space, with the three shocks (in this case) represented as dark lines.

A solution given by Theorem 2.1 shall be referred to, for brevity, as the *Oleinik solution to* $\{u_l, u_r\}$ centered on x_0.

2.1. The Generalized Riemann Problem for Initial Data With a Single Point of Discontinuity

Let $h(x_1, x_2, u_1, u_2; x) : R \to R$ be a continuous, differentiable, 1-1 mapping from the interval $[x_1, x_2]$ of the real axis onto a rarefaction interval $\{u_1, u_2\}$ of f such that

$$\begin{aligned} h(x_1, x_2, u_1, u_2; x_1) &= u_1 \ , \\ h(x_1, x_2, u_1, u_2; x_2) &= u_2 \ . \end{aligned} \tag{2.7}$$

Further, assume that the derivative of h is bounded and bounded away from 0 . Then its inverse map exists, is 1-1, onto, and has derivatives bounded away from 0. Denote the inverse map by

$$h^{-1}(x_1, x_2, u_1, u_2; u) \ .$$

Where no ambiguity exists the notation for these maps shall be shortened to $h(x)$ and $h^{-1}(u)$.

Let $f(u) \in C^2: R \to R$ such that $f''(u) = 0$ at a finite number of points. Let $\{a, b\}$ and $\{c, d\}$ be rarefaction intervals of $f(u)$.

Definition 2.2: The Cauchy problem

$$\frac{\partial u(x, t)}{\partial t} + \frac{\partial f(u(x, t))}{\partial x} = 0 , \tag{2.8}$$

with initial data

$$u(x) = \begin{cases} a & x \le x_l , \\ h_l(x_l, x_0, a, b; x) & x_l < x < x_0 , \\ h_r(x_0, x_r, c, d; x) & x_0 < x < x_r , \\ d & x_r \le x , \end{cases} \tag{2.9}$$

will be denoted as a Cauchy problem with generalized Riemann data (generalized Riemann problem, for short). The unique solution to (2.8) and (2.9) obeying (1.4) and (1.5) shall be referred to as the generalized Oleinik solution.

Theorem 2.3: Let $f(u) \in C^2: R \to R$ such that $f''(u) = 0$ for at most two values of u. Then the generalized Oleinik solution to (2.8), (2.9) is piecewise smooth; consists of two constant states, rarefaction fans, at most three shock waves[3] and at most two irregular points. A rarefaction wave either originates at $t = 0$ on the interval $[x_l, x_r]$ or tangentially to a shock line at $t > 0$ and either terminates on a shock line or propagates undisturbed for all t. At most two points can be irregular, the initial discontinuity in the data at $(t = 0, x = x_0)$ or the meeting point of three shocks[3].

2.1.1. Proof of Theorem 2.3

The following definitions will prove useful. Let $\{u_1, u_2\}$ be a rarefaction interval of f and $h(x_1, x_2, u_1, u_2; x)$ a map as in (2.7). At each point $x \in [x_1, x_2]$ construct the rarefaction wave $(u, 0, x)$ with

[3] By definition of a shock wave as a smooth curve, the occurences of either two shock lines interacting to produce a single shock curve, or a single shock curve splitting into two shock curves, are each described as involving three shock waves.

$$x(t) = x + f'(h(x))\, t \ .$$

The set of waves thus constructed on $[x_1, x_2]$ form a rarefaction fan which shall be referred to as the *rarefaction fan determined by* $h(x_1, x_2, u_1, u_2; x)$.

Let $\{u_1, u_2\}$ be a rarefaction interval of f. For every u in $\{u_1, u_2\}$ construct the rarefaction wave $(u, 0, x_0)$ with

$$x(t) = x_0 + f'(u)\, t \ .$$

The set of waves thus constructed form a rarefaction fan which shall be referred to as *determined at* x_0 *by the interval* $\{u_1, u_2\}$.

Let $\{a, b\}$ be a rarefaction interval of f. Let $h(\rho_0, \rho_1, a, b; \rho)$ be a map as in (2.7) from the real interval $[\rho_0, \rho_1]$ to $\{a, b\}$. Construct a curve segment $(t(\rho), x(\rho))$ from (t_0, x_0) to (t_1, x_1) in the t, x plane such that

$$(t_0, x_0) = (t(\rho_0), x(\rho_0)) \ ,$$

$$(t_1, x_1) = (t(\rho_1), x(\rho_1)) \ ,$$

$$dx(\rho)/dt(\rho) = f'(h(\rho_0, \rho_1, a, b; \rho)) \ .$$

At each point $(t(\rho), x(\rho))$ construct the rarefaction wave $(h(\rho), t(\rho), x(\rho))$. Then the set of waves constructed on the curve segment form a rarefaction fan which shall be referred to as the *rarefaction fan determined by the curve* $(t(\rho), x(\rho))$.

Lemma 2.4: Let $\mathcal{N}$ be a rarefaction fan in the t, x plane for some solution $u(t, x)$. Let $[x_1, x_2]$ be an interval in $\mathcal{N}$ parallel to the x-axis at some time t such that the rarefaction wave passing through the point (t, x_1) has value $u = u_1$ and the rarefaction wave passing through the point (t, x_2) has value $u = u_2$. Then $\{u_1, u_2\}$ is a rarefaction interval of f and the fan defines a continuous, 1-1 mapping $h'(x_1, x_2, u_1, u_2; x) : R \to R$ from $[x_1, x_2]$ onto $\{u_1, u_2\}$ whose derivatives are bounded and bounded away from 0 .

The proof for Lemma 2.4 follows from the definition of a rarefaction wave. □

The proof for Theorem 2.3 is constructive. All possible Cauchy data (2.9) fall into four classes:

C1: $\{a, b\}$ and $\{c, d\}$ are not disjoint, $f'' > 0$ on $\{a, b\} \cup \{c, d\}$,

C2: $\{a, b\}$ and $\{c, d\}$ are not disjoint, $f'' < 0$ on $\{a, b\} \cup \{c, d\}$,

C3: $\{a, b\}$ and $\{c, d\}$ are disjoint, $b > c$,

C4: $\{a, b\}$ and $\{c, d\}$ are disjoint, $b < c$.

Classes C3 and C4 can each be divided into four subgroups:

$$\begin{array}{ll} S1: & a < b, \quad c < d \ , \\ S2: & a > b, \quad c > d \ , \\ S3: & a < b, \quad c > d \ , \\ S4: & a > b, \quad c < d \ . \end{array}$$

For classes C1, C2 there are no inflection points of f in the interval $[a, d]$. For subgroups S1 and S2 of classes C3 and C4 there are either 0 or 2 inflection points in the interval $[a, d]$. Subgroups S3 and S4 of classes C3 and C4 contain one inflection point in the interval $[a, d]$. Theorem 2.3 will be proven for the case of 0 and 1 inflection point in the interval $[a, d]$ of f. The proof for the case of two inflection points will be omitted for brevity.

f has Zero Inflection Points

Restriction of f to no inflection points implies initial data from either classes C1, C2 or subgroups S1 or S2 of C3 and C4. We study data of classes C1, C3.S1 and C3.S2; the proofs for classes C2, C4.S1 and C4.S2 follow analogously.

Class C1

There are four arrangements of initial data in the class C1:

$$\begin{array}{ll} \text{C1.A1} & a < c < b < d \ , \\ \text{C1.A2} & a < c < d < b \ , \\ \text{C1.A3} & c < a < d < b \ , \\ \text{C1.A4} & c < a < b < d \ . \end{array}$$

The generalized Oleinik solution to Cauchy data of type C1.A1 is examined below.

Let $\{a, b\}$ and $\{c, d\}$ be rarefaction intervals of type C1.A1 . Consider maps $h_l(x_l, x_0, a, b; x)$ and $h_r(x_0, x_r, c, d; x)$ as in (2.7). Construct the rarefaction fans determined by $h_l(x)$ and $h_r(x)$. Then through each point $p = (t, x)$ in the plane such that

$$x_0 + f'(c)\, t \le x \le x_0 + f'(b)\, t$$

there passes a unique rarefaction wave originating from the interval $[x_l, x_0]$ and a unique rarefaction wave originating from the interval $[x_0, x_r]$, thus defining respectively the unique values $u_l(p)$ and $u_r(p)$.

Consider the curve $(t, x(t))$ defined implicitly by

$$x(t) = x_0 + \int_0^t \frac{dx}{dt}\, dt\ , \tag{2.10}$$

where

$$\frac{dx}{dt} = \frac{f(u_l(t)) - f(u_r(t))}{u_l(t) - u_r(t)}\ , \tag{2.11}$$

with $u_l(t)$ and $u_r(t)$ determined from

$$x(t) = h_l^{-1}\,(x_l, x_0, a, b; u_l(t)) + f'(u_l(t))\, t\ , \tag{2.12}$$

$$x(t) = h_r^{-1}\,(x_0, x_r, c, d; u_r(t)) + f'(u_r(t))\, t\ . \tag{2.13}$$

We remark that for $\{a, b\}$ and $\{c, d\}$ of type C1.A1, the set of equations (2.10) through (2.13) are well defined for all t and the curve $(t, x(t))$ is smooth.

Proposition 2.5: For the curve $(t, x(t))$ defined by (2.10) through (2.13):

1) $\dfrac{d\,u_l(t)}{dt} < 0\ ,\ \dfrac{d\,u_r(t)}{dt} > 0,\ \forall t > 0\ .$

2) There exists $g \in [c, b]$ such that $u_l(t) \to g\ ,\ u_r(t) \to g$ as $t \to \infty$.

3) The curve defined in (2.10) through (2.13) considered as a curve of points of jump $u_l(t) \to u_r(t)$ obeys (1.4) and (1.5).

Proof: Note

$$f'(u_l(t)) > 0, \quad (h_l^{-1}(u))' > 0, \quad \forall\, u_l(t) \text{ on } [a, b]\ ,$$

$$f'(u_r(t)) > 0, \quad (h_r^{-1}(u))' > 0, \quad \forall\, u_r(t) \text{ on } [c, d]\ .$$

Equations (2.12) and (2.13) yield

$$[f'(u_l(t)) - f'(u_r(t))]\, t = h_r^{-1}(u_r(t)) - h_l^{-1}(u_l(t))\ . \tag{2.14}$$

For $t > 0$, the positivity of the right hand side of (2.14) implies

$$f'(u_l(t)) - f'(u_r(t)) > 0\ . \tag{2.15}$$

By the requirement that $\{a, b\}$ and $\{c, d\}$ be of class C1.A1, (2.15) implies (using 2.11)

$$f'(u_l(t)) > \frac{dx}{dt} > f'(u_r(t)) \ . \tag{2.16}$$

Therefore, the time derivatives of (2.12) and (2.13),

$$\frac{dx}{dt} = [(h_l^{-1}(u_l))' + f''(u_l)\, t]\, u_l'(t) + f'(u_l) \ , \tag{2.17}$$

$$\frac{dx}{dt} = [(h_r^{-1}(u_r))' + f''(u_r)\, t]\, u_r'(t) + f'(u_r) \ , \tag{2.18}$$

lead to the conclusion

$$u_l'(t) < 0 \ , \ u_r'(t) > 0 \ , \ \forall t \ . \tag{2.19}$$

Further (2.16) and (2.19) imply the existence of $g \in [c, b]$ such that as $t \to \infty$,

$$u_l(t) \to g \ , \quad u_r(t) \to g \ , \quad \frac{dx}{dt} \to f'(g) \ .$$

By construction, the curve $(t, x(t))$ considered as a curve of points of jump obeys (1.4). It is easy to verify that the entropy condition (1.5) is also satisfied. From Proposition 2.5 note that the strength of the shock decreases monotonically to zero as $t \to \infty$. □

All rarefaction waves originating in the interval

$$h_l^{-1}(g) \le x \le h_r^{-1}(g)$$

for the construction in Proposition 2.5 intersect the curve $(t, x(t))$. Terminate these waves at the curve. All rarefaction waves originating in the intervals

$$h_l^{-1}(a) \le x \le h_l^{-1}(g)$$

$$h_r^{-1}(g) \le x \le h_r^{-1}(d)$$

propagate unimpeded for all $t > 0$. The rarefaction fans and shock curve thus defined by this construction shall be said to be *determined by the functions* $h_l(x_l, x_0, a, b; x)$ *and* $h_r(x_0, x_r, c, d; x)$.

Proposition 2.6: Let $\{a, b\}$ and $\{c, d\}$ be as in C1.A1 . Then the generalized Oleinik solution $u(t, x)$ to (2.8) (2.9) is given by

the constant state $u = a$ for $x \le x_l + f'(a)\, t\ ,\ \ t \ge 0\ ,$

the shock curve and rarefaction fans determined by $h_l(x)$ and $h_r(x)$ for $x_l + f'(a)\, t < x < x_r + f'(d)\, t\ ,\quad t \ge 0\ ,$

the constant state $u = d$ for $x_r + f'(d)\, t \le x\ ,\ \ t \ge 0\ ,$

The proof follows immediately from Proposition 2.5. □

A sketch of this solution is given in Fig. 2.2 .

The generalized Oleinik solution to Cauchy data of types C1.A2 through C1.A4 have exactly the same structure as for C1.A1 . For each, the strength of the single shock curve $(t, x(t))$ in each solution decreases monotonically to zero as $t \to \infty$, and $dx / dt \to f'(g)$ where for

C1.A2 $\quad c < g \le d\ ,$
C1.A3 $\quad a < g \le d\ ,$
C1.A4 $\quad a < g \le b\ .$

Class C3.S1 - No Inflection Point

Let $\{a, b\}$ and $\{c, d\}$ be rarefaction intervals of type C3.S1 such that for every $v \in \{a, b\},\ w \in \{c, d\}$

$$\mathcal{E}(v, w; u) = C(v, w; u)\ .$$

Then $f'(v) > f'(w)$ for every choice v, w. Consider the curve $(t, x(t))$ implicitly defined by (2.10) through (2.13). Statement 1 of Proposition 2.5 follows from (2.14) through (2.19). However, the disjointness of the intervals implies there exists finite times $t_l > 0,\ t_r > 0$ such that $u_l(t_l) = a$ and $u_r(t_r) = d$. For all $t > t_l$ let $u_l(t) = a$ and for all $t > t_r$ let $u_r(t) = d$. With this extension, (2.10) through (2.13) are defined for all t. The curve $(t, x(t))$ thus defined, considered as a curve of points of jump $u_l(t) \to u_r(t)$ obeys the conditions (1.4) and (1.5). In contrast to class C1 data, the shock strength attains a constant value, $|a - d|$, as $t \to \infty$.

All rarefaction waves originating from the interval $[x_l, x_r]$ intersect the shock curve and are terminated there. The shock curve and rarefaction fans so defined (together with the bounding constant states $u = a$, $u = d$ will also be referred to as *determined by the functions* $h_l(x)$ *and* $h_r(x)$ since their definition is merely an extension of the usage defined after Proposition 2.5.

Proposition 2.7: Let $\{a, b\}$ and $\{c, d\}$ be disjoint rarefaction intervals on f such that for all $v \in \{a, b\}$, $w \in \{c, d\}$

$$\mathcal{E}(v, w; u) = C(v, w; u) .$$

Then the generalized Oleinik solution to (2.8) (2.9) is given by constructing the shock curve and rarefaction fans determined by $h_l(x_l, x_0, a, b; x)$ and $h_r(x_0, x_r, c, d; x)$.

Proof: The above construction proves this proposition for rarefaction intervals in class C3.S1, where the condition $f'' > 0$ was used implicitly in (2.17) and (2.18). With trivial modifications, analogous arguments can be used to prove this proposition for appropriate data in classes C3 or C4.

Class C3.S2 - No Inflection Point

Data of class C3.S2 insures $f'(b) \leq f'(c)$.

Proposition 2.8: Let $\{a, b\}$ and $\{c, d\}$ be rarefaction intervals of f of class C3. Further assume $f'(b) \leq f'(c)$. Then the generalized Oleinik solution $u(t, x)$ to (2.8) (2.9) is

The constant state $u(t, x) = a$ for $x < x_l + f'(a)\, t$, $t \geq 0$,

The rarefaction wave determined by $h_l(x_l, x_0, a, b; x)$ for

$$x_l + f'(a)\, t < x < x_0 + f'(b)\, t\ ,\ \ t \geq 0\ ,$$

The Oleinik solution to $\{b, c\}$ centered on x_0 for $x_0 + f'(b)\, t < x < x_0 + f'(c)\, t$, $t \geq 0$,

The rarefaction wave determined by $h_r(x_0, x_r, c, d; x)$ for

$$x_0 + f'(c)\, t < x < x_r + f'(d)\, t\ ,\ \ t \geq 0\ ,$$

The constant state $u(t, x) = d$ for $x < x_r + f'(d)\, t$, $t \geq 0$.

Proof: The proof is trivial; the condition $f'(b) \leq f'(c)$ guarantees that the rarefaction waves determined by h_l and h_r do not interact.

Thus all solutions for allowed Cauchy data with f restricted to have no inflection point are piecewise smooth, with at most a single shock and at most a single irregular point at $(t = 0, x = x_0)$.

f has One Inflection Point

Restriction of f to a single inflection point implies initial data from subgroups S3 or S4 of C3 and C4. We study data of C3.S3 and C3.S4; the proofs for C4.S3 and C4.S4 follow analogously.

Class C3.S3

Lemma 2.9: Let f, $\{a, b\}$, and $\{b^*, a^*\}$ be chosen such that: $\{a, b\}$ is a rarefaction interval of f on which $f''(u) > 0$, $\{b^*, a^*\}$ is a rarefaction interval of f on which $f''(u) < 0$, $a^* < b$, and

$$\mathcal{E}(b, b^*; u) = C(b, b^*; u) , \qquad f'(b^*) = \frac{d\, C(b, b^*; u)}{du} ,$$

$$\mathcal{E}(a, a^*; u) = C(a, a^*; u) , \qquad f'(a^*) = \frac{d\, C(a, a^*; u)}{du} .$$

For every $v \in \{a, b\}$ there exists $w(v) \in \{b^*, a^*\}$ such that

$$\mathcal{E}(v, w(v); u) = C(v, w(v); u) , \qquad f'(w(v)) = \frac{d\, C(v, w(v); u)}{du} .$$

Then

1) $w(v)$ is a 1-1 map from $\{a, b\}$ onto $\{b^*, a^*\}$,

2) $f''(w(v))\dfrac{d\, w(v)}{dv} = \dfrac{f'(w(v)) - f'(v)}{w(v) - v}$.

Proof:

1) This follows from the definition of $w(v)$.

2) For all $v \in (a, b)$, δv sufficiently small,

$$f(w(v)) - f(v) = f'(w(v))\,[w(v) - v] , \tag{2.20}$$

$$f(w(v + \delta v)) - f(v + \delta v) = f'(w(v + \delta v))\,[w(v + \delta v) - v - \delta v] , \tag{2.21}$$

Subtracting (2.20) from (2.21), arranging terms and taking $\delta v \to 0$

$$f'(w(v))\,\frac{d\, w(v)}{dv} - f'(v) = f''(w(v))\,\frac{d\, w(v)}{dv}\,w(v) + f'(w(v))\,\frac{d\, w(v)}{dv} - f''(w(v))\,\frac{d\, w(v)}{dv}\,v - f'(w(v)) ,$$

from which the conclusion 2) follows. □

Let $\{a, b\}$ and $\{b^*, a^*\}$ be as in Lemma 2.9. Let $h_l(x_l, x_0, a, b; x)$ be the usual map from the rarefaction interval $\{a, b\}$ to $[x_l, x_0]$. Consider the curve $(t, x(t))$ defined by

$$x(t) = x_0 + \int_0^t \frac{dx}{dt}\, dt\,, \tag{2.10}$$

$$\frac{d\,x(t)}{dt} = f'(w(u_l(t)))\,, \tag{2.22}$$

with $w(u_l(t))$ defined in Lemma 2.9 and $u_l(t)$ defined by

$$x(t) = h_l^{-1}\,(x_l, x_0, a, b; u_l(t)) + f'(u_l(t))\,t\,. \tag{2.12}$$

There exists a finite time $t_l > 0$ at which $u_l(t_l) = a$ and

$$\frac{dx}{dt} = f'(w(a)) = f'(a^*).$$

For every $t > t_l$ let $u_l(t) = a$. Then (2.10), (2.12) and (2.22) are defined for all t.

Proposition 2.10: The curve defined by (2.10), (2.12) and (2.22) considered as a curve of points of jump $u_l \to w(u_l)$ obeys (1.4) and (1.5).

Proof: Follows from the construction. □

Construct the rarefaction fan determined by h_l. If a wave from this fan intersects the above shock curve, terminate the wave on the curve. For a point $(t_b, x(t_b))$ on this shock curve where $0 \le t_b \le t_l$, there exists a uniquely defined $u_r(t_b) = w(u_l(t_b))$. Also note from (2.22) and Lemma 2.9

$$x''(t) = f''(w(u_l))\, w'(u_l)\, u_l'(t) = \frac{f'(w(u_l)) - f'(u_l)}{w(u_l) - u_l}\, u_l'(t) < 0. \tag{2.23}$$

The rarefaction fan determined by this curve can therefore be constructed. Note further that the 'left bounding' wave $(a^*, t, x(t))$ of the fan propagates parallel to the shock curve.

The two rarefaction fans and shock wave, together with the bounding constant state $u = a$, thus constructed from (2.10), (2.12), (2.22) and (2.23) shall be denoted as *determined by the function $h_l(x)$ and the state $u = b^*$*.

Let f have a single inflection point and $\{a, b\}$ and $\{c, d\}$ be rarefaction intervals of type C3.S3 . Then there exists a rarefaction interval $\{b^*, a^*\}$ as described in Lemma 2.9. Three possibilities exist for the orientation of $\{c, d\}$ with respect to $\{b^*, a^*\}$:

Case 1: $\{c, d\}$ and $\{b^*, a^*\}$ are disjoint with $d \ge a^*$.

Case 2: $\{c, d\}$ and $\{b^*, a^*\}$ are disjoint with $c \le b^*$.

Case 3: $\{c, d\}$ and $\{b^*, a^*\}$ are not disjoint.

For the case $d \geq a^*$, the generalized Oleinik solution $u(t, x)$ to (2.8) (2.9) is given by Proposition 2.7 .

For the case $c \leq b^*$, the generalized Oleinik solution $u(t, x)$ to (2.8) (2.9) is given by the following:

The constant state $u = a$, two rarefaction fans and the shock curve determined by the function $h_l(x)$ and the state $u = b^*$ for $x \leq f'(b^*)\, t$, $t > 0$.

The rarefaction wave determined at x_0 by the interval $\{c, b^*\}$ for $f'(b^*)\, t \leq x \leq f'(c)\, t$, $t > 0$.

The rarefaction wave determined by the $h_r(x)$ for $f'(c)\, t \leq x \leq f'(d)\, t$, $t > 0$.

The constant state $u = d$ for $f'(c)\, t \leq x$, $t > 0$.

This solution is sketched in Fig. 2.3 .

For the case in which $\{c, d\}$ and $\{b^*, a^*\}$ are not disjoint we remark that the generalized Oleinik solution can be described by a solution similar to case 1 above for $0 \leq t \leq t_g$ and by a solution similar to case 2 for $t_g < t$. The time t_g is determined by the propagation of a rarefaction wave for some state $u = g$ where $g \in \{b^*, a^*\} \cap \{c, d\}$.

Class C3.S4

Lemma 2.9 and Proposition 2.10 can be written analogously for data of type C3.S4, with the arguments revolving around an interval $\{c^*, d^*\}$. Again the construction of a solution to the generalized Riemann problem can be broken into three cases and dealt with as above.

The solutions for classes C4.S3 and C4.S4 follow in the same manner. Thus we have shown that the solutions to the generalized Riemann problem for f having at most a single inflection point are piecewise smooth and consist of at most a single shock wave and at most a single irregular point (at x_0).

f has Two Inflection Points

In the interest of brevity, we omit the solution for the case of f having two inflection points. The method of proof is repetitive to that presented above. The proof shows that in addition to

solutions with a single shock, either two shock waves can be present for all $t > 0$ or three shock waves can appear in the solution [i.e. a single shock which is present for $0 \le t \le t_s$ which then 'splits' (via interaction with rarefaction waves) into two shocks which are present for $t_s \le t$].

2.2. A Conjecture on Loss of Piecewise Smoothness for More Than Two Inflection Points

The constructive proof given for Theorem 2.3 is not readily extendable to f having three inflection points since the number of cases to be considered is large. In fact, we argue that for $f(u)$ having three or more inflection points, the solution need not be piecewise smooth. An example is sketched below in which a solution having a countably infinite number of arbitrarily small smooth pieces may occur.

Let $\{a, b\}$ and $\{c, d\}$ be rarefaction intervals of $f(u)$ as illustrated in the lower left hand side of Fig. 2.4 . $f(u)$ has three inflection points in the interval $[d, b]$. Consider the corresponding generalized Riemann problem (2.8), (2.9). Using the construction methods of the proof of Theorem 2.3, it is seen that the generalized Oleinik solution initially has a single shock curve determined by $h_l(x)$ and $h_r(x)$ starting at x_0 with slope $d\,C(b, c; u) / du$. The rates at which the left and right states $u_l(t)$ and $u_r(t)$ change along the shock curve depend on the functions $h_l(x)$ and $h_r(x)$. Given $\{c, d\}$, (and hence $\{a, b\}$), $[x_0, x_r]$ and $h_r(x)$, it is possible to determine $h_l(x)$ and $[x_l, x_r]$ such that for all time $t > 0$

$$\mathcal{E}(u_l(t), u_r(t); u) = C(u_l(t), g(t); u) \cup C(g(t), u_r(t); u) , \tag{2.24}$$

where

$$f'(g(t)) = \frac{\partial C(u_l(t), g(t); u)}{\partial t} = \frac{\partial C(g(t), u_r(t); u)}{\partial t} . \tag{2.25}$$

The convex envelope (2.24), though consisting of two chords, defines a single shock line and represents a metastable balance between a single and a two shock configuration. This construction can be perturbed to create a single 'bubble'; that is a time interval $t_1 < t < t_2$ over which the shock separates into two shocks as is illustrated on the right in Fig. 2.4 . This is accomplished by adjustment of the rate at which the rarefaction waves (i.e. adjustment of the respective h functions) enter the shock waves from the extreme left and right. The above construction can be iterated to produce a second bubble starting at t_3 by adjusting (slowing) the rate $h_l'(x)$

appropriately. It has not as yet been determined if, for a given set of intervals $\{a, b\}$, $\{c, d\}$, $[x_l, x_0]$, and $[x_0, x_r]$, it is possible to construct a sequence of generalized Riemann problems, where the i'th problem is characterized by the choice of functions $h_l^i(x)$, $h_r^i(x)$, for which the solution to the i'th problem has i such bubbles. If this was possible, the solution to the limiting problem obtained as $i \to \infty$ would not be piecewise smooth. We note it is certainly possible to create an infinite sequence of such problems if only $\{a, b\}$, $\{c, d\}$, and $[x_r, x_0]$ are held constant and the interval $[x_0, x_l]$ is allowed to vary with i. It remains to be proven that such a sequence of problems can be chosen such that $[x_0, x_l]$ also remains fixed.

3. The Generalized Riemann Problem for Initial Data with Multiple Points of Discontinuity

Let $x_l < x_0 < x_1 < \cdots < x_n < x_r$. Consider (2.8) with initial data

$$u(0, x) = \begin{cases} u_l & x < x_l \,, \\ h_l(x_l, x_0, u_l, u_0; x) & x_l < x < x_0 \,, \\ h_j(x_j, x_{j+1}, u_j, u_{j+1}; x) & x_j < x < x_{j+1} \,, \quad j = 0, 1, \cdots, n-1 \,, \\ h_r(x_n, x_r, u_n, u_r; x) & x_n < x < x_r \,, \\ u_r & x_l \le x \,, \end{cases} \tag{3.1}$$

where $h_l(x)$, $h_j(x)$, $h_r(x)$ are as in (2.7), and $\{u_l, u_0\}$, $\{u_0, u_1\}$, ..., $\{u_n, u_r\}$ are rarefaction intervals of f. The Cauchy problem (2.8), (3.1) is an extension of the definition of the generalized Riemann problem to include multiple initial discontinuities.

Theorem 3.1: The solution to (2.8) with initial data (3.1) for $f \in C^2 : R \to R$ having at most one inflection point is piecewise smooth. There are no more than $(n+1)!$ shocks in the solution and no more than $(n+1)!$ irregular points.

The restriction to either zero or one inflection point for $f(u)$ guarantees that when two shocks interact a single shock is produced. Thus given a finite number m of initial discontinuities, the number of shocks in the solution remains bounded by $m!$ and the solution will be piecewise smooth. For f having two or more inflection points there is no a priori bound on the growth of the number of shocks in time since an interaction of two (incoming) shocks can conceivably give rise to two or more (outgoing) shocks (or to a single shock which later splits via interaction with rarefaction waves into two or more shocks) and the possibility of generating a solution that is richer in structure than piecewise smooth exists. It is only through the decrease in variation of the

solution (ie. the decrease in shock strength with in time) that such a bound may be possible. Such a bound on the growth rate of shock number combined with a general statement for the > 2 inflection point case of §2 is required for a more general statement on the piecewise smoothness of the solution to the generalized Riemann problem for initial data with multiple points of discontinuity.

Theorem 3.1 will be proven by finite induction on n in (3.1). The case $n = 0$ is the generalized Riemann of §2.1 . The case $n = 1$ is given below. It suffices to illustrate the salient points of the general induction step.

Proof of Theorem 3.1 for $n = 1$: Consider equation (2.8) under three possible cases of Cauchy data:

$$u(x) = \begin{cases} u_l & x \le x_l \,, \\ h_l(x_l, x_0, u_l, u_0; x) & x_l < x < x_0 \,, \\ h_0(x_0, x_1, u_0, u_1; x) & x_0 < x < x_1 \,, \\ u_1 & x_1 \le x \,, \end{cases} \tag{3.2}$$

$$u(x) = \begin{cases} u_0 & x \le x_0 \,, \\ h_0(x_0, x_1, u_0, u_1; x) & x_0 < x < x_1 \,, \\ h_r(x_1, x_r, u_1, u_r; x) & x_1 < x < x_r \,, \\ u_r & x_r \le x \,, \end{cases} \tag{3.3}$$

$$u(x) = \begin{cases} u_l & x \le x_l \,, \\ h_l(x_l, x_0, u_l, u_0; x) & x_l < x < x_0 \,, \\ h_0(x_0, x_1, u_0, u_1; x) & x_0 < x < x_1 \,, \\ h_r(x_1, x_r, u_1, u_r; x) & x_1 < x < x_r \,, \\ u_r & x_r \le x \,. \end{cases} \tag{3.4}$$

From §2, construct the solutions to (3.2) and (3.3), denoting them $u_L(t, x)$ and $u_R(t, x)$ respectively. Denote the solution to (3.4) as $u_C(t, x)$. The following four (exhaustive but not mutually exclusive) possibilities exist for the solutions u_L and u_R.[4]

[4] As x_0 and x_1 are spatially separated, intuitively u_C is expected to be composed thusly:

u_C = left piece of u_L ∪ piece due interaction of waves x_0 and x_1 ∪ right piece of u_R .

Conditions P1 through P4 characterize the interactions between waves from x_0 and x_1 .

P1) There exists a rarefaction wave $(v, 0, x_v)$ common to both solutions with $v \in \{u_0, u_1\}$, $x_v \in (x_0, x_1)$ which propagates unimpeded for all t.

P2) Both u_L and u_R have a shock curve and there exists a point $p_2 = (t_2, x_2)$ common to the shock curve of each solution for some $t_2 > 0$.

P3) u_L has a shock curve, denoted $(u_l^1(t), u_r^1(t), x)$ and $u_r^1(t_3) = u_1$ for some $t_3 > 0$.

P4) u_R has a shock curve, denoted $(u_l^2(t), u_r^2(t), x)$ and $u_r^2(t_4) = u_0$ for some $t_4 > 0$.

If P1 holds (if neither solution has a shock wave, then condition P1 holds) then

$$u_C(x, t) = \begin{cases} u_L(x, t) & \text{if } x \le x_v + f'(v)\, t\,, \\ u_R(x, t) & \text{if } x > x_v + f'(v)\, t\,, \end{cases} \tag{3.5}$$

is the unique solution to (2.8) (3.1) obeying (1.4) and (1.5).

If P1 does not hold let $t_{\min}$ denote the minimum of the existing values t_2, t_3, t_4 .

If $t_{\min} = t_2$ then let $\Sigma_L^0 = (u_l^0(t), u_r^0(t), 0, x_0)$ and $\Sigma_R^1 = (u_l^1(t), u_r^1(t), 0, x_1)$ denote the shock waves propagating respectively from x_0 in u_L and x_1 in u_R. Then for $t \le t_2$

$$u_C(x, t) = \begin{cases} u_L(x, t) & \text{if } x \le h_0^{-1}(v_2) + f'(v_2)\, t\,, \\ u_R(x, t) & \text{if } x > h_0^{-1}(v_2) + f'(v_2)\, t\,, \end{cases} \tag{3.6}$$

where $v_2 = u_r^0(t_2) = u_l^1(t_2)$ is in the interval $\{u_0, u_1\}$. For $t > t_2$, $u_C(x, t)$ is the generalized Oleinik solution to (2.8) with initial data (at t_2)

$$u(t_2, x) = \begin{cases} u_l & x \le x_l^{t_2}\,, \\ h_l^{t_2}(x_l^{t_2}, x_2, u_l, u_l^1(t_2); x) & x_l^{t_2} < x < x_2\,, \\ h_r^{t_2}(x_2, x_r^{t_2}, u_r^2(t_2), u_r; x) & x_2 < x < x_r^{t_2}\,, \\ u_r & x_r^{t_2} \le x\,, \end{cases}$$

where

$$x_l^{t_2} = x_l + f'(u_l)\, t_2\,,$$
$$x_r^{t_2} = x_r + f'(u_r)\, t_2\,.$$

$h_i^{t_2}(x)$ is defined by the rarefaction interval $\{u_l, u_l^1(t_2)\}$ where $u_l^1(t_2)$ is the left limit of u at (t_2, x_2) for the shock curve Σ_0 in u_C propagating from x_0 in (3.6). (Σ_0 is identical with Σ_L^0 for $0 \le t \le t_2$.) Similarly $h_r^{t_2}(x)$ is defined by the rarefaction interval $\{u_r, u_r^2(t_2)\}$ where $u_r^2(t_2)$ is the

right limit of u at (t_2, x_2) for the shock curve Σ_1 propagating from x_1. (Σ_1 is identical with Σ_L^1 for $0 \le t \le t_2$.)

If $t_{\min} = t_3$, let $x_3 = x_1 + f'(u_1)\, t$. There exist two possibilities for the solution u_C if u_R has a shock wave. (If u_R has no shock the solution for u_C is deduced easily from the discussion below.) These two possibilities are either that u_R has a leftmost rarefaction fan

a) centered at the point x_1 (Fig. 3.1a),

b) determined by the shock wave Σ_1 originating at $(0, x_1)$ (Fig. 3.1b).

We note that in either case this leftmost rarefaction fan in u_R is a smooth continuation of the rightmost rarefaction fan in the solution u_L for $t \le t_3$. For $t \le t_3$, let

$$u_C(t, x) = \begin{cases} u_L(t, x), & x \le x_1 + f'(u_1)\, t, \\ u_R(t, x), & x \ge x_1 + f'(u_1)\, t. \end{cases} \tag{3.7}$$

At (t_3, x_3) the shock Σ_0 originating from $(0, x_0)$ encounters the first rarefaction wave from the solution for $x > x_3$. For case a) above, the shock Σ_0 is determined for $t > t_3$ by the rarefaction fan determined by $h_l(x_l, x_0, u_l, u_0; x)$ and the rarefaction fan centered on $(0, x_1)$. Denote the curve followed by the shock Σ_0 for $t > t_3$ as $(t, x_{\Sigma_0}(t))$. Then, for case a),

$$u_C(t, x) = \begin{cases} u_L & x < x_{\Sigma_0}(t), \\ u_R & x > x_{\Sigma_0}(t), \end{cases} \qquad t > t_3. \tag{3.8}$$

In case b) the shock Σ_0 is determined for $t > t_3$ by the rarefaction fan determined by $h_l(x_l, x_0, u_l, u_0; x)$ and the rarefaction fan determined by the shock line Σ_1. In either case a) or b), the resulting shock, Σ_0 hits the shock Σ_1 at some time $t_5 > t_3$ or it does not. If it does not, the solution for all $t > t_3$ is given by (3.8). If it does, the solution for $t_3 \le t \le t_5$ is given by (3.8) and for $t \ge t_5$ is given by the solution to a generalized Riemann problem of the type discussed in §2.1. It is to be noted that the solution is C^1 continuous at t_3 (ie. all waves propagate continuously from $t_3 - \epsilon$ to $t_3 + \epsilon$ and all waves except the interacting shock lines are C^1 continuous at t_5.

If $t_{\min} = t_4$, an analogous construction to that for case P3 holds.

In all cases the constructed solution u_C is thus piecewise smooth. The general induction step for Theorem 3.1 follows the above argument with a more complicated notation. □

4. The 2-Dimensional Riemann Problem

For convenience we shall use the notation $f_1 = f$, $f_2 = g$, $x_1 = x$, $x_2 = y$ for the discussion in two spatial dimensions.

Definition 4.1: The Cauchy problem (1.1) with initial data piecewise constant on a finite number of wedges focused on a single point in the x, y plane shall be referred to as the *two dimensional Riemann problem*. Without loss of generality, this point can be taken to be the point $x = 0, y = 0$.

For the case $g = f$, under the 45° rotation $2\xi = x + y$, $2\eta = y - x$, (1.1) becomes

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial \xi} = 0 . \tag{4.1}$$

From (4.1) we see that the solution can be obtained along each η = *constant* plane independent of other η. In particular, this implies the solutions obtained in the $\eta < 0$ half space can be obtained independently of the solutions in the $\eta > 0$ half; by the symmetry of the problem, no new solutions will be found in the $\eta < 0$ half space that are not found in the upper. We therefore restrict our discussion to the half space $\eta > 0$ ($y > x$). The solution in the plane $\eta = 0$ is discussed in Lemma 4.4 .

In the $\eta > 0$ half space, the solution in each η = *constant* plane is just that for a generalized Riemann problem in one dimension.

Lemma 4.2: Let $u(t, \xi, \eta_0)$ denote the generalized Riemann problem solution in the $\eta = \eta_0 > 0$ plane. Then the solution $u(t, \xi, \eta_1)$ to the generalized Riemann problem in the η_1 plane ($\eta_1 > 0$) is just a similarity solution of $u(t, \xi, \eta_0)$ given by

$$u(c\,t, c\,\xi + b(c), \eta_1) = u(t, \xi, \eta_0) , \tag{4.2}$$

where $c = \dfrac{\eta_1}{\eta_0}$.

Proof: If $v_0(\xi)$ is the initial data for the generalized Riemann problem on η_0, then $w_0(\zeta = c\,\xi + b(c)) = v_0(\xi)$ is the initial data for the generalized Riemann problem on η_1 where

$c = \frac{\eta_1}{\eta_0}$ and $b(c)$ is given by the wedge pattern of the two dimensional Riemann problem. Further, the function $u(\tau = ct, \zeta = c\,\xi + b, \eta_1 = c\,\eta_0) = u(t, \xi, \eta_0)$ is a weak solution of the generalized Riemann problem on η_1 having initial data w_0 . □

Thus the solution to the 2-dimensional Riemann problem (4.1) in the $\eta > 0$ half space is found from a mapping M: $R^1 \to R^2$ which is a similarity transformation in t, ξ, η and therefore continuous onto the domain $t > 0, \xi, \eta > 0$.

Theorem 4.3: The unique (in the sense of Kruzkov) solution in the plane $\eta > 0$ ($y > x$), to (1.1) with initial data that is piecewise constant on a finite number of wedges focused on a single point in the plane, with $f_1 = f_2 = f, f \in C^2 : R \to R, f$ having at most one inflection point, is

a) piecewise smooth.

b) composed of non-linear waves and constant states which are the images under a similarity map M defined by Lemma 4.2 of the rarefaction and shock waves and constant states of a generalized Riemann problem in one dimension.

c) composed of curves of irregular points corresponding to images under the map M of the irregular points in the 1-dimensional Riemann problem.

Proof: Follows immediately from Lemma 4.2 and Theorem 3.1 . □

Lemma 4.4:

a) If the half line $\eta = 0, \xi < 0$ is not a line of discontinuity of the initial data (ie: is not a wedge line) the waves incident upon the corresponding half plane $t > 0, \xi < 0, \eta = 0$ are continuous across the half plane.

b) If the half line $\eta = 0, \xi < 0$ is a line of discontinuity (constant jump) of the initial data, it remains a half plane $t > 0, \xi < 0, \eta = 0$ of (in general variable) jump discontinuity in the solution.

Proof:

a) Follows from taking the limit $\eta \to 0$ of the solution in either half plane.

b) Follows from taking the limit of solutions for which the wedge line in the initial data is displaced slightly from the $\eta = 0$ axis. □

Similar statements hold for the half line $\eta = 0, \xi > 0$.

Although the map M provides a means of constructing the solution to the 2-dimensional Riemann problem, it is not a convenient method of doing so. In particular it is a method that will not generalize to cases $f \neq g$ where no appeal to a one dimensional analysis can be made. We propose that the correct method of dealing with the general solution to the 2-dimensional Riemann problem is to identify the general two dimensional non-linear waves. In [4] we identify these waves for the case $f = g$ and use the construction method initiated by Guckenheimer [1] and Wagner [6] to piece these waves together into entropy obeying solutions.

5. Concluding Remarks

The condition $f_1 = f_2 = f$, $f \in C^2 : R \to R$, f having at most one inflection point has been shown to be sufficient to guarantee that the solution to the 2-dimensional Riemann problem is piecewise smooth. A construction has been presented detailing a presumed mechanism for loss of piecwise smoothness if f has three or more inflection points.

The Riemann problem in one dimension has been generalized to include a finite number of discontinuities and smooth 'rarefaction' variation in the initial data. Sufficiency conditions based upon the number of inflection points in the convection function f have been obtained which guarantee piecwise smoothness of the solution to the generalized problem.

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Figure Captions

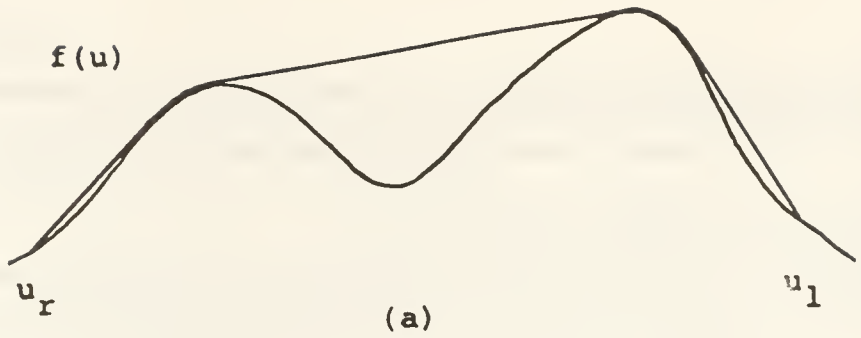
Fig. 2.1 a) The upper convex envelope drawn between two points u_l and u_r for an illustrative function f. b) The Riemann solution in the t, x plane based on a). Dark lines represent shock waves, light lines are rarefaction waves.

Fig. 2.2 A sketch of the solution (lower figure) for a particular case of initial data (upper figure) for a generalized Riemann problem with f having no inflection point. The dark line is the shock wave. Important rarefaction lines have been labelled by their speeds.

Fig. 2.3 A sketch of the solution (lower figure) for a particular case of initial data (upper figure) for a generalized Riemann problem with f having a single inflection point. The dark line is the shock wave. Important rarefaction lines have been labelled by their speeds.

Fig. 2.3 Formation of a single 'bubble' in a solution. The dotted lines in the middle diagrams on the right emphasise the non - linearity of the convex envelope.

Fig. 3.1 Two possibilities encountered under case P3 described in the text. Dark lines are shock waves, light lines are rarefaction waves.



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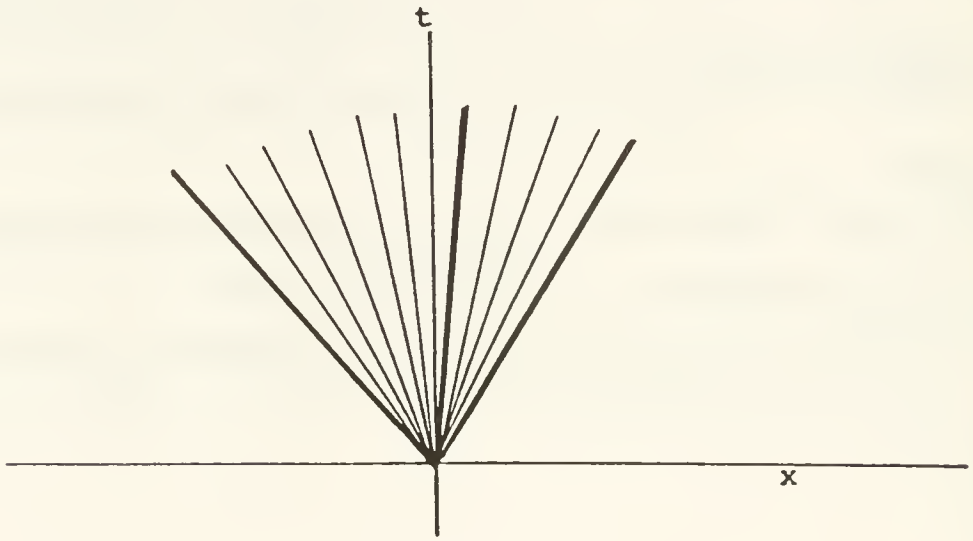


(b)

Fig. 2.1

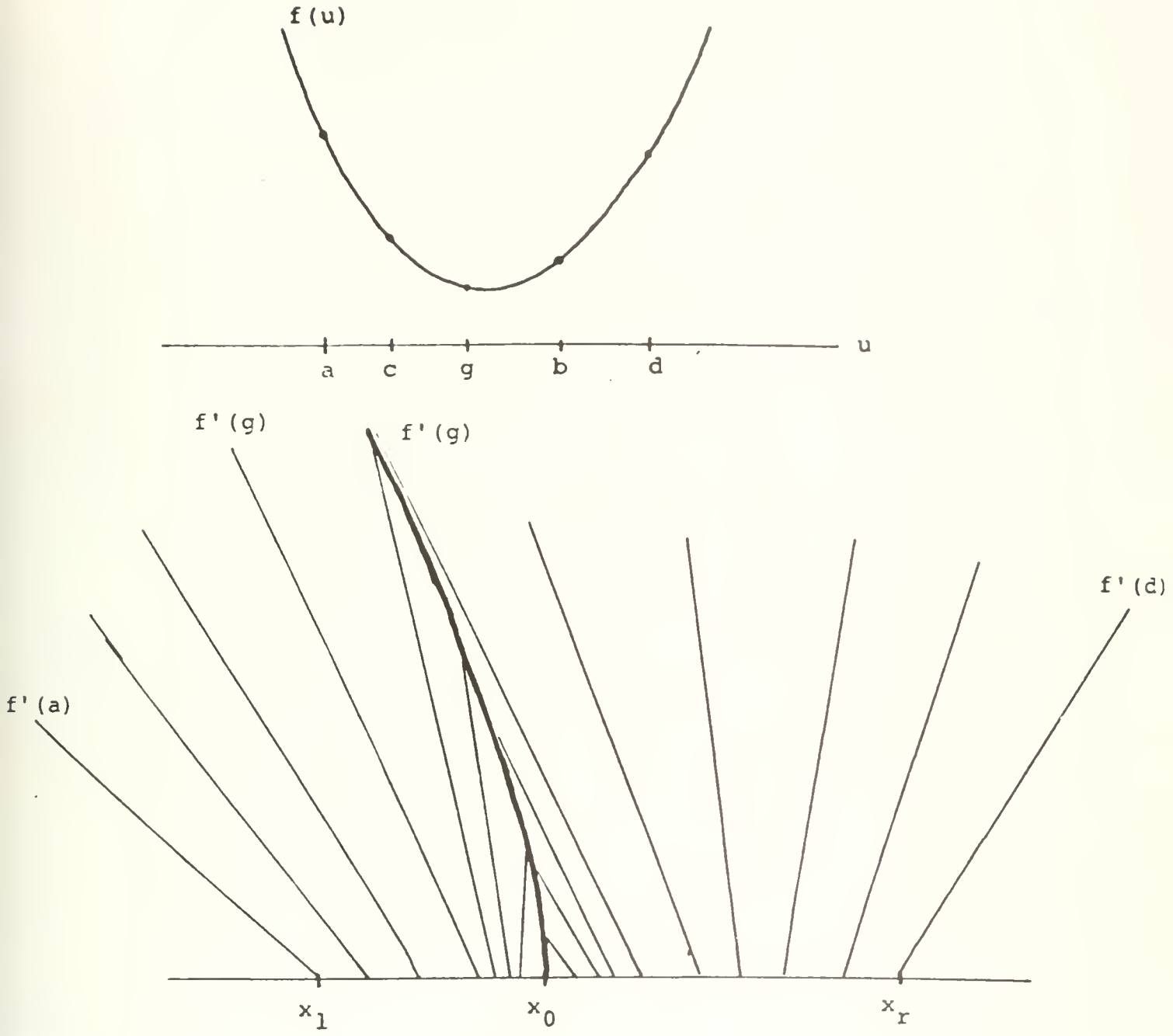


Fig. 2.2

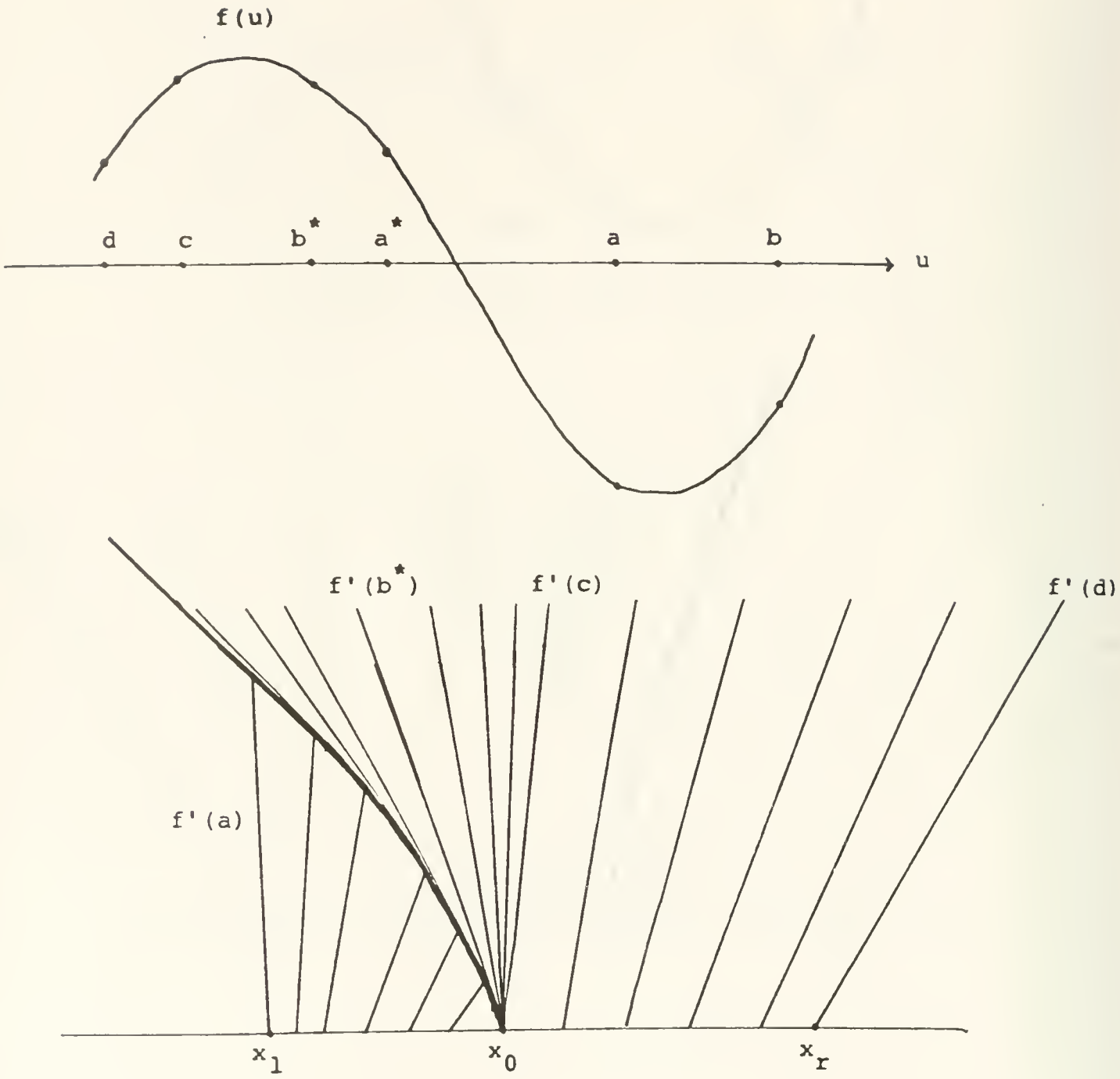


Fig. 2.3

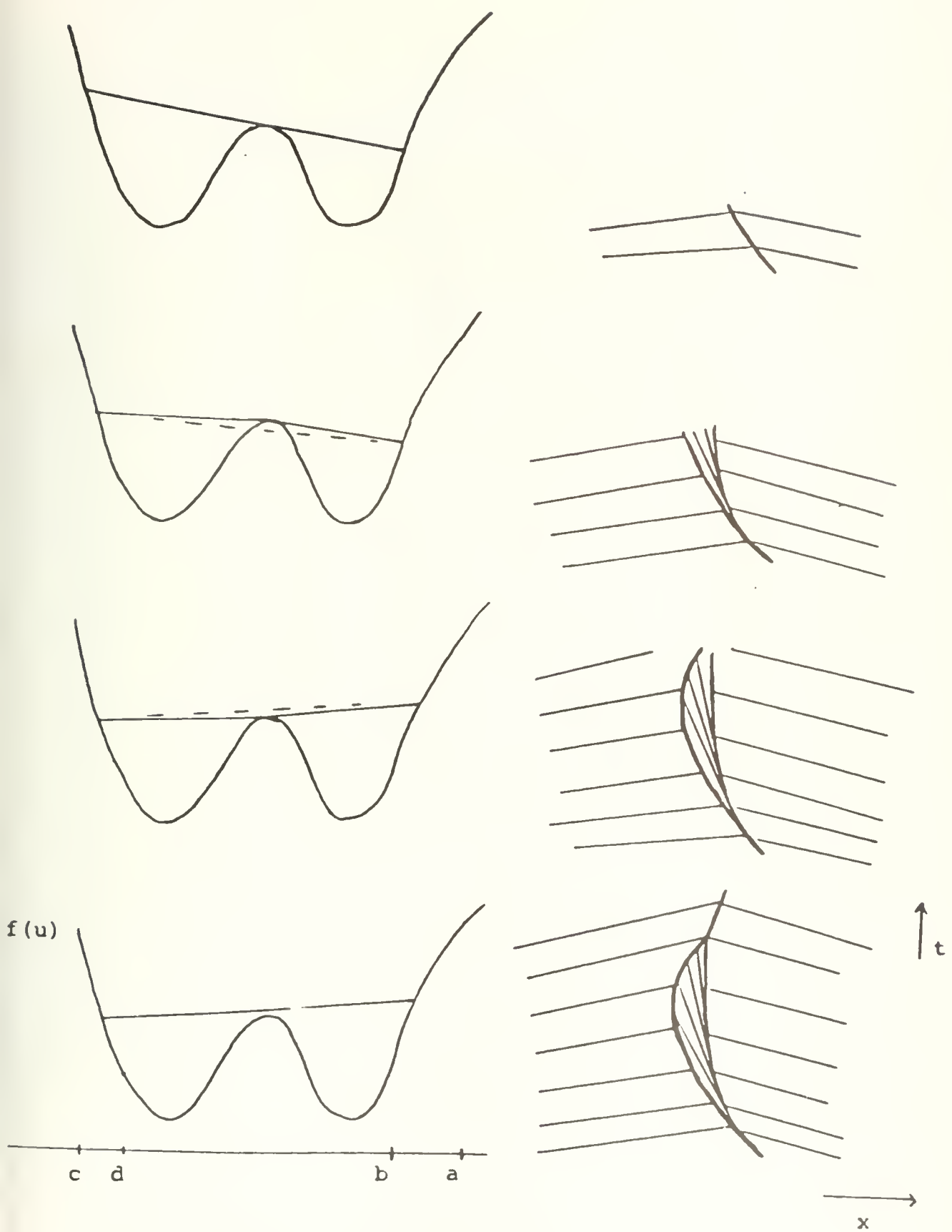
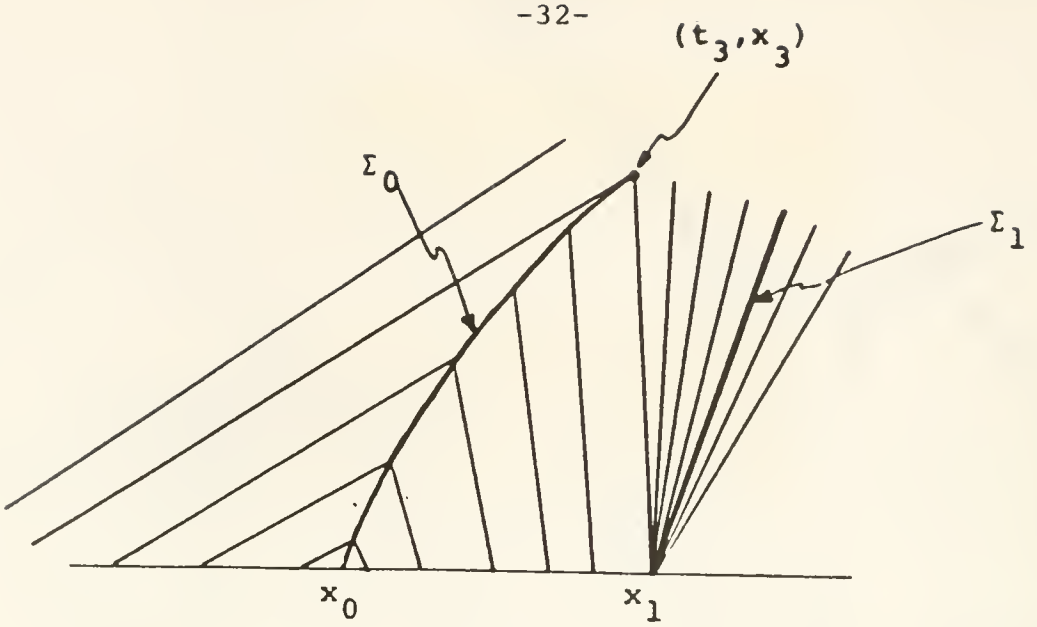
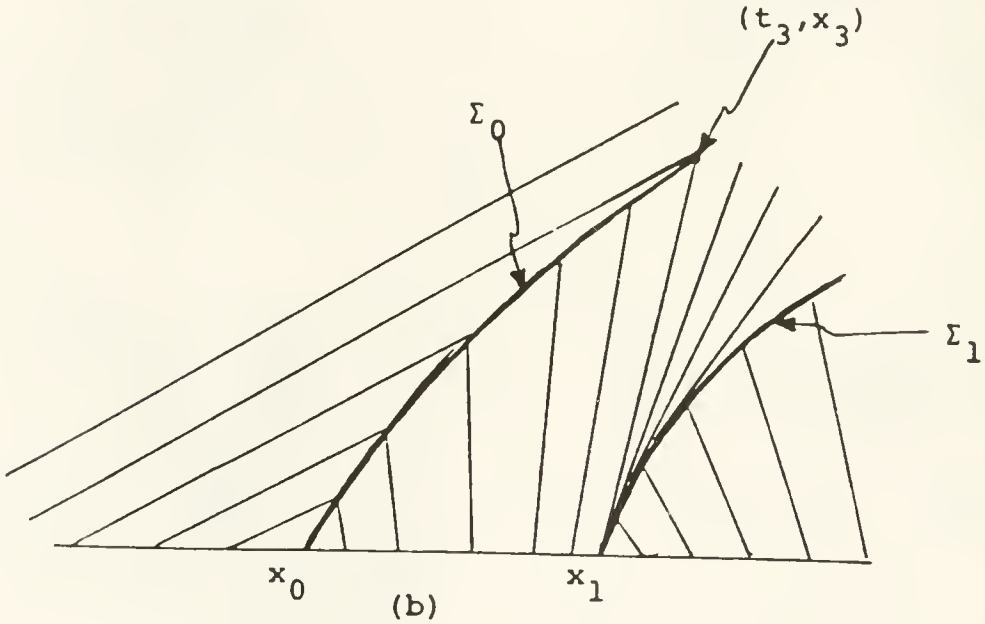


Fig. 2.4



(a)



(b)

Fig. 3.1

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